Problem Set 6

Macroeconomics III

Max Blichfeldt Ørsnes

Department of Economics
University of Copenhagen

Fall 2021

Problem 1

We consider an OLG model with the following characteristics:

- Constant population - each generation consists of N people.
- Endowment when young: y_1
- Endowment when old: 0

It is not possible save money for future periods.

The utility function is given by:

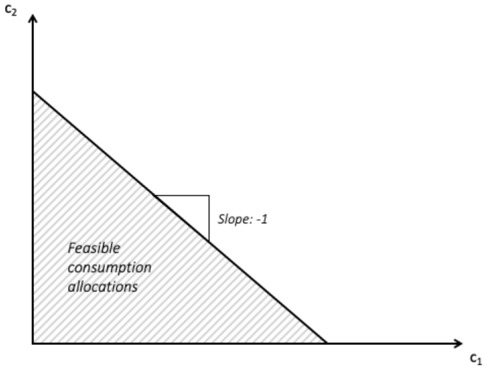
$$U_t = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

Problem 1a - Feasible consumption

Define a feasible consumption allocation for this economy. Illustrate the set of feasible allocations on a graph.

The resource constraint/feasibility constraint is given by:

$$y_1 \geq c_{1t} + c_{2t}$$



Problem 1b - Pareto efficiency

Define a Pareto effcient stationary allocation for this economy.

A pareto efficient allocation is an allocation such that no agent in the economy can be made better off without making another agent worse off.

The allocation is stationary when $c_{1t} = c_1$ and $c_{2t} = c_2$.

Problem 1c - Pareto efficiency in this economy

Solve for the Pareto effcient stationary allocation.

We maximize the total welfare of the agents s.t. the resource constraint.

$$\max_{c_1, c_2} U = \ln(c_1) + \beta \ln(c_2) \qquad \text{s.t. } y_1 \leq c_1 + c_2$$

Since utility is strictly increasing in c_1 and c_2, we know that the resource constraint holds with equality: $y_1 = c_1 + c_2$.

We rewrite the problem:

$$\max_{c_1} U = \ln(c_1) + \beta \ln(y_1 - c_1)$$

Which leads to the following FOC::

$$\frac{1}{c_1^*} = \frac{\beta}{y_1 - c_1^*} \quad \Longrightarrow \quad c_1^* = \frac{y_1 - c_1^*}{\beta} \quad \Longrightarrow \quad c_1^* = \frac{1}{1+\beta} y_1$$

From the resource constraint we get:

$$c_2 = y_1 - c_1 \quad \Longrightarrow \quad c_2^* = \frac{\beta}{1+\beta} y_1$$

Problem 1d and 1e - Competitive equilibrium

d) Define a competitive equilibrium without money for this economy. Will there be any trades between individuals in the economy? Explain.

e) Solve for the consumption allocation in the competitive equilibrium without money. Is the allocation the same as in c)? Which consumption allocation does each generation prefer?

The young generation would like to smooth consumption but have no savings method. Lending money to the old is not attractive, since the current old are dead in the following period and the young cannot make a contract with the future young.

No trading will trade place and the competitive allocation will be $(c_1, c_2) = (y_1, 0)$.

Which consumption bundle do they prefer?

All generations prefer (c_1^*, c_2^*). The allocation $(y_1, 0)$ was a feasible bundle in question c, so (c_1^*, c_2^*) is clearly superior. $u(c_1^*, c_2^*) > u(y_1, o)$.

Problem 1f - Introducing fiat money

Now suppose that each member of the initial old generation is endowed with m_0 units of fiat money (the total stock of fiat money is $M = Nm_0$). Define a competitive equilibrium with money for this economy.

The new constraints for the household are now:

$$m_t^d = P_t(y_1 - c_{1t}) \tag{1}$$

$$m_t^d = P_{t+1} c_{2t+1} \tag{2}$$

Where (1) is the money demand of the young for the goods they didn't consume and (2) is the money demand for future consumption.

The competitive equilibrium is then characterized by:

- Optimizing behaviour of households:

$$\max_{c_{1t}, c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1}) \qquad \text{s.t. (1) and (2)}$$

- Markets clear: Money supply equals money demand, $m_0 = m_t^d$

Problem 1g - Competitive eq. with fiat money (1/2)

Solve for the consumption allocation and the demand for real balances, $\frac{M_t^d}{P_t}$, in the competitive equilibrium with money as a function of the rate of return of fiat money, $\frac{P_t}{P_{t+1}}$.

The household problem is given by the utility function and the budget constraints from problem 1f:

$$\max_{c_{1t}, c_{2t+1}} \quad U_t = \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = y_1 - \frac{m_t^d}{P_t}$$

$$c_{2t+1} = \frac{m_t^d}{P_{t+1}}$$

Substituting the budget constraints into the utility funciton yields:

$$\max_{m_t^d} U_t = \ln\left(y_1 - \frac{m_t^d}{P_t}\right) + \beta \ln\left(\frac{m_t^d}{P_{t+1}}\right)$$

Problem 1g - Competitive eq. with fiat money (2/2)

$$U_t = \ln\left(y_1 - \frac{m_t^d}{P_t}\right) + \beta \ln\left(\frac{m_t^d}{P_{t+1}}\right)$$

Maximizing wrt. m_t^d yields:

$$\frac{\beta}{m_t^d} = \frac{1}{P_t}\frac{1}{y_1 - \frac{m_t^d}{P_t}}$$

$$\frac{m_t^d}{\beta} = P_t\left(y_1 - \frac{m_t^d}{P_t}\right)$$

$$\frac{m_t^d}{P_t} = \beta y_1 - \beta\frac{m_t^d}{P_t}$$

$$\frac{m_t^d}{P_t}(1+\beta) = \beta y_1$$

$$\frac{m_t^d}{P_t} = \frac{\beta}{1+\beta}y_1 \qquad (3)$$

Inserting (3) into the budget constraints yields:

$$c_{1t} = y_1 - \frac{m_t^d}{P_t}$$

$$c_{1t} = y_1 - \frac{\beta}{1+\beta}y_1$$

$$c_{1t} = \frac{1}{1+\beta}y_1 \qquad (4)$$

$$c_{2t+1} = \frac{m_t^d}{P_{t+1}}$$

$$c_{2t+1} = \frac{P_t}{P_{t+1}}\frac{m_t^d}{P_t}$$

$$c_{2t+1} = \frac{P_t}{P_{t+1}}\frac{\beta}{1+\beta}y_1 \qquad (5)$$

Problem 1h - Return on money and consumption in eq.

Find the rate of return of fiat money and the consumption allocation in the stationary competitive equilibrium with money.

The demand for real money is $\frac{\beta}{1+\beta}y_1$ and thus time invariant. Hence,

$$\frac{m_t^d}{P_t} = \frac{m_{t+1}^d}{P_{t+1}} = \frac{\beta}{1+\beta}y_1$$

Furthermore, money markets must clear such that:

$$m_0 = m_t^d = m_{t+1}^d$$

Combining these leads to:

$$P_t = P_{t+1}$$

Which means that the rate of return of fiat money is equal to 1. Consumption then becomes:

$$c_{1t} = \frac{1}{1+\beta}y_1 \qquad c_{2t+1} = \frac{\beta}{1+\beta}y_1$$

Thus, the competitive equilibrium is the pareto efficient allocation when fiat money is introduced.

Problem 2 - OLG w. money growth + lump-sum transfers

Explain the reason why in the OLG model with money it is not possible for the equilibrium with money growth and lump sum transfers to Pareto dominate the monetary equilibrium with constant money stock. In other words, why the equilibrium with money growth is not to the northeast of point J in graph 4.4 (page 162 of BF)?

The money in itself does not add any utility to the economy. The value of money in this economy exists because money adds a savings method - a way to distribute the endowment across generations.

The money does not change the feasibility constraint: $y_1 \geq c_{1t} + c_{2t}$. Since the pareto efficient allocation was achieved by introduced a fixed amount of money, adding money growth does imply a pareto improvement.

Problem 3

We consider an open OLG economy with population growth $L_{t+1} = (1+n)L_t$.

Households solve the following problem

$$
\begin{aligned}
\max_{c_{1t}, c_{2t+1}} \quad & U_t = \ln(c_{1t}) + \beta \ln(c_{2t+1}) \\
\text{s.t.} \quad & c_{1t} = w_t - s_t \\
& c_{2t+1} = R s_t
\end{aligned}
$$

The economy is initially closed and in a steady state where: $R^A < 1+n$.

Note: It is dynamic inefficient.

The international interest rate is $R^W < R^A$.

Furthermore, we have the usual results for wages and interest rates:

$$
R = 1 + r \qquad \text{where} \qquad
\begin{aligned}
r &= f'(k) \\
w &= f(k) - f'(k)k
\end{aligned}
$$

Problem 3a - Maximization problem + indirect utility

Show that opening to trade makes everyone worse off. Interpret.

The households optimize such that:

$$c^*_{1t} = \frac{1}{1+\beta} w_t \qquad c^*_{2t+1} = \frac{\beta}{1+\beta} R w_t$$

We then insert the optimal consumption into the utility function because we want to investigate how a change in R affects the utility.

$$\begin{aligned} U^*_t &= \ln(c^*_{1t}) + \beta \ln(c^*_{2t+1}) \\ &= \ln\left(\frac{1}{1+\beta} w_t\right) + \beta \ln\left(\frac{\beta}{1+\beta} R w_t\right) \\ &= \ln(w_t) - \ln(1+\beta) + \beta \ln(R) + \beta \ln(w_t) - \beta \ln(1+\beta) + \beta \ln(\beta) \\ &= (1+\beta)\ln(w_t) + \beta \ln(R) + \underbrace{\beta \ln(\beta) - (1+\beta)\ln(1+\beta)}_{X} \end{aligned}$$

Hence, the indirect utility is given by:

$$U^* = (1+\beta)\ln(w) + \beta \ln(R) + X$$

Problem 3a - Derivative of indirect utility (1/3)

$$U^* = (1+\beta)\ln(w) + \beta\ln(1+r) + X$$

We find the derivative of U^* wrt. R, since the interest rate will become $R^W < R^A$ if the economy opens.

$$\frac{\partial U^*}{\partial r} = \frac{(1+\beta)}{w}\frac{\partial w}{\partial r} + \frac{\beta}{(1+r)}$$

A decrease in R lowers the return on capital but increases wages. We know that:

$$w = f(k) - f'(k)k \qquad \text{and} \qquad r = f'(k)$$

Taking derivative wrt. r:

$$\frac{\partial w}{\partial r} = f'(k)\frac{\partial k}{\partial r} - f''(k)\frac{\partial k}{\partial r}k - f'(k)\frac{\partial k}{\partial r} = -f''(k)\frac{\partial k}{\partial r}k$$

And

$$1 = f''(k)\frac{\partial k}{\partial r} \qquad \Longrightarrow \qquad \frac{\partial w}{\partial r} = -k$$

Problem 3a - Derivative of indirect utility (2/3)

Now we have the following:

$$\frac{\partial w}{\partial r} = -k$$
$$\frac{\partial U_t^*}{\partial r} = \frac{\beta}{(1+r)} + \frac{(1+\beta)}{w}\frac{\partial w}{\partial r}$$

We insert the derivative of w into the derivative of the indirect utility.

$$\frac{\partial U_t^*}{\partial r} = \frac{\beta}{1+r} - k\frac{(1+\beta)}{w}$$

Finally, we look at the capital accumulation in a closed economy to get an expression for k:

$$k_{t+1} = \frac{s_t}{1+n} = \frac{1}{1+n}\frac{\beta}{1+\beta}w_t$$

Problem 3a - Derivative of indirect utility (3/3)

We insert this in the derivative of the indirect utility.

$$\frac{\partial U}{\partial r} = \frac{\beta}{R} - k\frac{(1+\beta)}{w} = \frac{\beta}{1+r} - \underbrace{\frac{1}{1+n}\frac{\beta}{1+\beta}w}_{=k}\frac{1+\beta}{w}$$

$$= \frac{\beta}{1+r} - \frac{\beta}{1+n}$$

We can see that:

$$\frac{\partial U}{\partial r} > 0 \quad \text{for} \quad r < n \qquad \text{and} \qquad \frac{\partial U}{\partial r} < 0 \quad \text{for} \quad r > n$$

We know that $r < n$, why the derivative is positive. Hence, a decrease in the interest rate decreases utility in this economy.

Was this as expected?

Yes. The economy was dynamically inefficient, which means that the interest rate is inefficiently low. Opening the economy lowers the interest rate further, why it exacerbates the problem of dynamic inefficiency.

Problem 3b - A sudden increase in r

Suppose that after the economy is open it reaches a steady state before t. Show that a small permanent increase in the interest rate that occurs on period t benefits not only the period t old, but also the young born in t and every period afterwards. Interpret.

The young: We have just shown that an increase in the interest rate will increase the utility, so they will obviously benefit.

The old: They benefit because they get a higher return on their savings, which increases their disposable income.

Future: Everyone will benefit from the permanent increase.

Note: If the increase in the interest rate was very large such that $r^{NEW} > n$, then the analysis would be less obvious.